C2M10

Sequences

The concept of *limit* is as basic as it gets in calculus. And, to understand how a limit works one must learn to recognize two ingredients, *accuracy* and *control*. Intuitively, when our input gets close to one value our output will get close to another, which we call the limit. So, we must have a means of discussing "getting close to". A *sequence* is a function whose domain is the natural numbers, or a subset thereof, and whose range is the real numbers, $\mathbb{R}$. We can also have sequences of integrals, matrices, or complex numbers, but real numbers will suffice for now.

Notation: $\{a_n\} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ denotes a sequence whose n^{th} member is a_n.

Imagine a sequence as a succession of projectiles directed towards the bulls-eye of a target, where the center of the bulls-eye is regarded as the limit. We adjust our aim so that the rest of the projectiles will be striking within one inch of that center. The accuracy is the one inch, and that must precede the control. How do we invoke a control? From some projectile on, the rest of them will be withing the requested accuracy. Let's formalize the concept of the limit of a sequence.

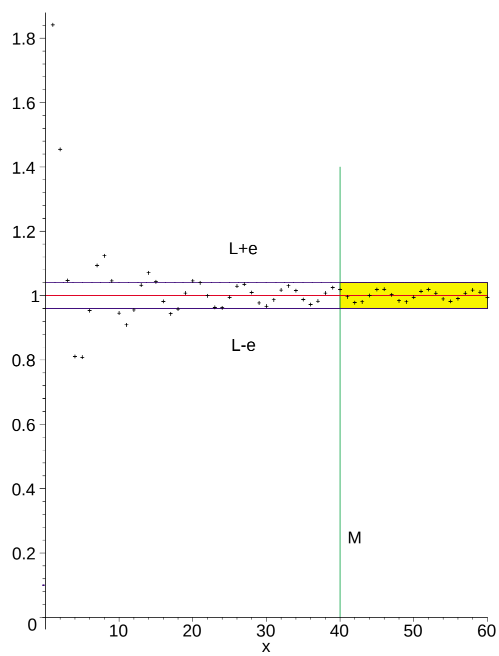
Definition: A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \ \text{ as } \ n \to \infty$$

if, for each $\epsilon > 0$ (accuracy) there is a number M (control) so that

$$n > M \ \Rightarrow \ |a_n - L| < \epsilon$$

If we invoke the control by having n larger than M, then our sequence member a_n will be within our predetermined accuracy, $\epsilon > 0$.



In the diagram above, all members of the sequence to the right of the vertical line marked M lie in the shaded region between the two horizontal lines at $L - e$ and $L + e$. That is what is meant by '$|a_n - L| < \epsilon$ for $n > M$'.

When your instructor told you that a sequence is a function whose domain is the natural numbers, or a subset thereof, it is possible that you did not attach as much importance to that idea as you did to the mechanics of dealing with sequences. Maple allows us to define expressions and functions, and it is sometimes confusing as to which we want to use. We will define the sequence $\{a_n\} = \left\{\dfrac{n+2}{3n-1}\right\}$ as an expression and the sequence $\{b_n\} = \left\{\sqrt{n^2+3n} - n\right\}$ as a function to illustrate how they must be handled differently.

Maple Example 1: $\{a_n\} = \left\{\dfrac{n+2}{3n-1}\right\}$ Use Maple to determine the limit when the sequence is defined as an expression.

```
> with(student):
> a:=(n+2)/(3*n-1);
```

$$a := \frac{n+2}{3n-1}$$

```
> seq(a,n=1..10);
```

$$\frac{3}{2}, \frac{4}{5}, \frac{5}{8}, \frac{6}{11}, \frac{1}{2}, \frac{8}{17}, \frac{9}{20}, \frac{10}{23}, \frac{11}{26}, \frac{12}{29}$$

```
> limit(a,n=infinity);
```

$$\frac{1}{3}$$

Maple Example 2: $\{b_n\} = \left\{\sqrt{n^2+3n} - n\right\}$ Use Maple to find the limit when the sequence is defined as a function.

```
> restart:    with(student):
> b:=n->sqrt(n^2+3n)-n;
```

$$b := n \to \sqrt{n^2+3n} - n$$

```
> seq(b(n),n=1..10);
```

$$1, \sqrt{10}-2, 3\sqrt{2}-3, 2\sqrt{7}-4, 2\sqrt{10}-5, 3\sqrt{6}-6, \sqrt{70}-7, s\sqrt{22}-8, 6\sqrt{3}-9, \sqrt{130}-10$$

```
> limit(b(n),n=infinity);
```

$$\frac{3}{2}$$

To understand this last limit, consider multiplying b_n by its conjugate, and then dividing by it.

$$\left(\sqrt{n^2+3n} - n\right) \cdot \frac{\sqrt{n^2+3n}+n}{\sqrt{n^2+3n}+n} = \frac{n^2+3n-n^2}{\sqrt{n^2+3n}+n} = \frac{3n}{\sqrt{n^2+3n}+n} \cdot \frac{1/n}{1/n} = \frac{3}{\sqrt{1+3/n}+1} \to \frac{3}{2}$$

with the limit taken as $n \to \infty$.

The sequence $\{a_n\}$ is obtained from an expression whose name is a while the sequence $\{b_n\}$ is obtained by evaluating a function whose name is b. If we had the command `seq(b,n=1..10);` what would we have obtained? The answer - ten b's, because the function b must be evaluated in order for it to have a value. This will be very important in the next section when we will need to consider the term $\dfrac{b_{n+1}}{b_n} = \dfrac{b(n+1)}{b(n)}$. It would be cumbersome and less clear to find $\dfrac{a_{n+1}}{a_n}$ when a is an expression. The command would be `subs(n=n+1,a)/a;`.

C2M10 Problems Using Maple, find the first ten terms of each sequence and the limit of each.

1. $a_n = \left\{\dfrac{n^2}{3^n}\right\}$
2. $b_n = \left\{\dfrac{n^2-3n+4}{5+2n+6n^2}\right\}$
3. $c_n = \left\{\left(1+\dfrac{2}{n}\right)^n\right\}$
4. $d_n = \left\{\left(1-\dfrac{2}{n}\right)^n\right\}$
5. $e_n = \left\{\sqrt{n^2+6n} - n\right\}$